Orcina

Equation of Motion for Spar Buoys in OrcaFlex

This document describes the equation of motion that OrcaFlex builds and solves to model a spar buoy. The basic equation of motion for any OrcaFlex object is Newtons' Second Law:

$$[L]=[M].[A]$$

where L is the total load vector acting on the object, M is the total mass matrix and A is the buoy acceleration vector.

The spar buoy is assumed to be a rigid body and so it has 6 degrees of freedom, so both L and A are six-dimensional vectors and M is a 6x6 matrix. The acceleration vector A is the unknown variable, whereas L and M can be calculated and must be assembled by adding together their constituent parts. These are listed below.

All the vectors and matrices described in this document can be expressed with respect to either buoy axes or global axes. The documentation below defines them with respect to whichever is the most convenient set of axes. However when the vectors are added together to give the total load vector they are first re-oriented to be with respect to buoy axes.

How the Load Vector L and Mass Matrix M are Built

The load vector L and mass matrix M are zeroised and then contributions from the various effects present are added into them.

Load Vector

The total load vector L is a 6-dimensional vector that is made up of two 3-dimensional parts - namely the total force vector and the total moment vector. Each load contribution described below contributes a force F and moment H acting at some position P (relative to buoy origin), and the resulting contribution to L is $\begin{bmatrix} F \\ H + P \times F \end{bmatrix}$. (The extra moment contribution P x F is the moment of F about the buoy origin.) Note that in some of the contributions, only F is non-zero and H is zero.

Mass Matrix

A mass matrix is a 6x6 matrix that can be viewed as 2x2 block matrix $\begin{bmatrix} M_{TT} & M_{TR} \\ M_{RT} & M_{RR} \end{bmatrix}$ where each block is a 3x3 matrix. Here M_{TT} is the translation-translation block, M_{TR} is the translation-rotation block, M_{RT} is the rotation-translation block and M_{RR} is the rotation-rotation block.

If one of the effects described below makes a contribution $\begin{bmatrix} M_{TT} & M_{TR} \\ M_{RT} & M_{RR} \end{bmatrix}$ at the buoy origin, then that contribution is simply added into the overall buoy mass matrix M. However if an effect makes this contribution at an offset position P = (x,y,z) (with respect to buoy axes), then the offset must be taken into account. This result of taking this offset into account is that the following mass matrix is added into the overall mass matrix M of the buoy.

$$\begin{bmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -z & +y \\ +z & 0 & -x \\ -y & +x & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{bmatrix} \begin{bmatrix} M_{TT} & M_{TR} \\ M_{RT} & M_{RR} \end{bmatrix} \begin{bmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & +z & -y \\ -z & 0 & +x \\ +y & -x & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{bmatrix}$$

This effect of this formula is to take into account the moment arm effects of the offset at which the mass matrix is being added in.

Contributions to the Load Vector L and Mass Matrix M

We now list the various effects that make contributions to the buoy's load vector and mass matrix.

Weight and Structural Inertia

Weight load $F = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix}, H = 0$ (with respect to global axes) is applied at the buoy centre of mass. Here m is the buoy mass, as specified on the buoy data form, and g is the acceleration due to gravity.

Structural mass matrix $\begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 0 \\ 0 & 0 & 0 & I_x & 0 & 0 \\ 0 & 0 & 0 & 0 & I_y & 0 \\ 0 & 0 & 0 & 0 & 0 & I_z \end{bmatrix}$ (with respect to buoy axes) is applied at the buoy centre of mass, where I_x, I_y, I_z are the buoy mass moments of inertia, as specified on the buoy data form.

Buoyancy Force

For each wetted cylinder, buoyancy load $F = \begin{bmatrix} 0 \\ 0 \\ \rho V_w g \end{bmatrix}, H = 0$ (with respect to global axes) is applied at point P = instantaneous centroid of the wetted volume of the cylinder. Here:

- ρ is the water density
- V_w is the instantaneous wetted volume.

Note that V_w, and its centroid, are calculated by computing the volume of the cylinder when it is truncated by the water surface tangent plane (we have derived an analytic formula for this volume and its centroid). The tangent plane used depends on whether the added mass and damping is specified by "Values for each cylinder" or by "RAOs and matrices for Buoy". In the former case the water surface is assumed to be the tangent plane to the wave surface directly above the centre of the cylinder; this therefore allows for the buoyancy variation due to the wave. In the latter case the still water surface is used (rather than the instantaneous water surface), since the buoyancy variation due to the wave is assumed to be included in the wave force and moment RAOs specified (see "First Order Wave Effects" below).

First Order Wave Effects

The following first order effects are included only if the added mass and damping is specified by "RAOs and matrices for Buoy" (on the Added Mass and Damping tab on the buoy data form). In this case the cylinder damping loads and cylinder added mass effects described later are **not** included, since it is assumed that those effects are included in these first order wave effects. However the cylinder drag loads are still included, since these are a second order effect.

- The buoy added mass matrix (with respect to buoy axes), as specified on the buoy data form, is added in at the RAO, added mass and damping origin.

- First order wave force and moment (applied at the RAO, added mass and damping origin), as defined by the RAOs specified on the buoy data form. The RAOs specify the amplitude and phase lag of the surge, heave and pitch load components, relative to the amplitude and phase of the wave at the instantaneous position of the RAO, added mass and damping origin. Note that the surge and heave directions here are the x and z-directions of a right-handed set of axes that has z vertically upwards, x pointing in the horizontal direction in which the wave is progressing and y horizontal and normal to the wave direction. Positive surge and heave forces are forces in those positive x and z-directions, and a positive pitch moment is a clockwise moment about the y-direction. The sway, roll and yaw RAOs (with respect to this frame of reference) are assumed to be zero on symmetry grounds.

- Damping load $\begin{bmatrix} F \\ H \end{bmatrix} = [D].\begin{bmatrix} U_c - U_b \\ -W_b \end{bmatrix}$ (with respect to buoy axes), applied at the RAO, added mass and damping origin. Here:

    - D is the 6x6 damping matrix specified on the buoy data form.

    - If damping relative to fluid has been specified, then U_c is the current velocity at the RAO, added mass and damping origin. But if damping relative to earth has been specified, then U_c is zero.

    - U_b is the velocity of the RAO, added mass and damping origin and W_b is the angular velocity of the buoy.

    - Note that this damping load omits any contribution due to the wave since it is assumed that the damping load due to the wave is included in the wave force and moment RAOs.

Cylinder Drag Loads

For each wetted cylinder, drag load

$$F = \frac{V_w}{V} \times \begin{bmatrix} \frac{1}{2}\rho C_{dn} A_n U_x |U_n| \\ \frac{1}{2}\rho C_{dn} A_n U_y |U_n| \\ \frac{1}{2}\rho C_{da} A_a U_a |U_a| \end{bmatrix}, \quad H = \frac{V_w}{V} \times \begin{bmatrix} \frac{1}{2}\rho C_{mn} B_n W_x |W_n| \\ \frac{1}{2}\rho C_{mn} B_n W_y |W_n| \\ \frac{1}{2}\rho C_{ma} B_a W_a |W_a| \end{bmatrix} \quad \text{(with respect to buoy axes)}$$

is applied at point P = centroid of the wetted volume of the cylinder, where:

- V is the total volume of the cylinder. V_w/V is therefore the proportion wet.
- C_{dn} and C_{da} are the normal and axial drag force coefficients specified for the cylinder on the buoy data form.
- A_n and A_a are the normal and axial drag areas, as specified for the cylinder on the buoy data form.
- U_x, U_y, U_n and U_a are the components, in the buoy x-, y-, normal and axial directions respectively, of the relative velocity vector U, which equals the fluid velocity vector minus the velocity vector of the wetted centroid of the cylinder.
- C_{mn} and C_{ma} are the normal and axial drag moment coefficients, as specified for the cylinder on the buoy data form.
- B_n and B_a are the normal and axial drag area moments, as specified for the cylinder on the buoy data form.
- W_x, W_y, W_n and W_a are the components, in the buoy x-, y-, normal and axial directions respectively, of the relative angular velocity vector W, which is the angular velocity of the local water isobar minus the angular velocity vector of the cylinder.

Note that the buoy z-direction is along the buoy axis and the buoy x- and y-directions are normal to the buoy axis, so $U_n = \sqrt{U_x^2 + U_y^2}$, $U_a = U_z$ = z-component of U, and similarly for W. The above formulae for F and H are based on the "cross-flow" principle.

Cylinder Damping Loads

Note that the following cylinder damping loads are included only if the added mass and damping is specified by "Values for each cylinder ". If the added mass and damping is specified by "RAOs and matrices for buoy" then these cylinder damping loads are **not** included, since the effect is then assumed to be included in the wave force and moment RAOs specified.

For each wetted cylinder, damping load $F = \frac{V_w}{V} \times \begin{bmatrix} F_{un}U_x \\ F_{un}U_y \\ F_{ua}U_z \end{bmatrix}$, $H = \frac{V_w}{V} \times \begin{bmatrix} H_{un}W_x \\ H_{un}W_y \\ H_{ua}W_z \end{bmatrix}$ (with respect to buoy axes) is applied at point P = centroid of the wetted volume of the cylinder, where:

- F_{un} and F_{ua} are the cylinder's unit damping force values in the normal and axial direction, as specified on the buoy data form.

- U_x, U_y, U_z are the components, in the buoy x-, y- and z-directions respectively, of the relative velocity vector U (see "cylinder drag loads" above).

- H_{un} and H_{ua} are the cylinder's unit damping moment values in the normal and axial directions respectively, as specified on the buoy data form.

- W_x, W_y, W_z are the components, in the buoy x-, y- and z-directions respectively, of the relative angular velocity vector W (see "cylinder drag loads" above).

Cylinder Added Mass Effects

Note that the following cylinder added mass effects are included only if the added mass and damping is specified by "Values for each cylinder". If the added mass and damping is specified by "RAOs and matrices for buoy" then these cylinder added mass effects are **not** included, since the effect is then assumed to be included in the wave force and moment RAOs specified.

Note also that the following applies only to the case of constant added mass coefficients. The calculation for variable added mass coefficients is described fully in the OrcaFlex help file.

For each wetted cylinder, the following added mass matrix is applied at point P = centroid of the wetted volume of the cylinder:

$$\frac{V_W}{V} \times \begin{bmatrix} \begin{pmatrix} C_{an}\rho V & 0 & 0 \\ 0 & C_{an}\rho V & 0 \\ 0 & 0 & C_{aa}\rho V \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} I_{an} & 0 & 0 \\ 0 & I_{an} & 0 \\ 0 & 0 & I_{aa} \end{pmatrix} \end{bmatrix}$$ (with respect to buoy axes).

Here:

- C_{an} and C_{aa} are the cylinder's normal and axial added mass force coefficients, respectively, as specified on the buoy data form.

- I_{an} and I_{aa} are the cylinder's added moments of inertia about the normal and axial directions respectively, as specified on the buoy data form.

In addition, for each wetted cylinder, the following fluid acceleration load is applied at point P = centroid of the wetted volume of the cylinder:

$$F = \frac{V_w}{V} \times \begin{bmatrix} \rho V(1+C_{an})Q_x \\ \rho V(1+C_{an})Q_y \\ \rho V(1+C_{aa})Q_z \end{bmatrix},\ H = \frac{V_w}{V} \times \begin{bmatrix} (I_{dn}+I_{an})R_x \\ (I_{dn}+I_{an})R_y \\ (I_{da}+I_{aa})R_z \end{bmatrix} \text{ (with respect to buoy axes).}$$

Here:

- Q_x, Q_y and Q_z are the components, in the buoy x-, y- and z-directions respectively, of the fluid acceleration vector Q at the centroid of the wetted volume of the cylinder.

- I_{dn} and I_{da} are the displaced moments of inertia, about the normal and axial directions respectively. These are the moments of inertia of a cylinder of sea water of the same size as the cylinder.

- R_x, R_y and R_z are the components, in the buoy x-, y- and z-directions respectively, of the angular acceleration of the local water isobar at the wetted centroid of the cylinder.

Note that the "1" part of the "$1+C_{an}$" terms in the above formula for F is the Froude-Krylov force and the "C_{an}" part is sometimes called the added mass force (and similarly for the "$1+C_{aa}$" term). The formula for H is based on the same principles as F, so the "I_{dn}" part of the "$I_{dn} + I_{an}$" terms play the role of a Froude-Krylov moment and the "I_{an}" part plays the role of an added mass moment (and similarly for the "$I_{da} + I_{aa}$" term).

Note also that if the cylinder's added moment of inertia, I_{an}, is zero then the displaced moment of inertia I_{dn} term is also omitted; similarly if I_{aa} is zero then the I_{da} term is also omitted. This enables the user to deliberately omit all rotational fluid acceleration loads, by setting the added moments of inertia of the cylinder to zero.

Loads From Wings

If any wings are specified, then the forces and moments from those wings are applied at point P = the point of connection of the wing. Wings make no contribution to the buoy mass matrix.

Loads and Mass contributions from Connected Objects

If another object (e.g. a line, link or winch) is connected to the buoy, then it contributes a force, moment and mass matrix to the buoy. For example, if a line is connected to the buoy then the mass matrix of the end node is added into the buoy mass matrix and the line end force F and moment H are applied; all of these line contributions are applied at point P = the point of connection.

Applied Loads

The Applied Loads specified on the buoy data form are added into the buoy load vector L. They are applied at position P = the specified point of application.

Munk Moment

A Munk moment is applied only if the Munk moment coefficient specified on the buoy data form is non-zero. For details see the OrcaFlex help file.

Contact Forces

Contact forces are applied if the buoy comes into contact with the seabed or any solids in the model. These are applied at position P = the point of contact.

Slam loads

Slam loads may also be calculated and applied to a spar buoy, usually (but not necessarily) as it passes through the water surface. For details of their calculation, please see the OrcaFlex help file.

Orcina Ltd.
Daltongate
Ulverston
Cumbria LA12 7AJ
UK
Tel: +44 (0)1229 584742
Email: orcina@orcina.com